Gravity assist in 3D like in Ulysses mission

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Abstract

We study the gravity assist in the general case, i.e. when the spacecraft is not in a coplanar motion with respect to the planet's orbit. Our derivation is based on Kepler's planetary motion and Galilean addition of velocities, subjects covered in introductory physics courses. The main purpose of this paper is to illustrate how the gravity assist can be used to deviate a spacecraft outside its original plane of motion. As an example, we use the NASA-ESA's Ulysses mission to "test" our simple model.

I. INTRODUCTION

It is well known that gravity assist is an excellent technique to speed up or slow down a spacecraft, but is also a mechanism to deviate its course. For these reasons, NASA and ESA use the gravity assist as a helpful tool to reduce mission cost and to save fuel and travel time. Otherwise, planet exploration would be hard to pursue using current technologies.

This phenomenon, also called slingshot effect or swing-by maneuver, can be easily understood as an elastic collision of two particles.[1,2,3] However, there is a paradox in the elastic derivation, because if one assumes that the planet's mass is much greater than the spacecraft's, one would conclude that the initial and final speeds of the spacecraft (both when it is very far from the planet) should be equal, like in the usual Kepler motion. The "mistake" with this thinking is that one is forgetting the reference frame, since the above conclusion is correct for an observer on the planet, but not for an observer on the Sun due to the relative motion between the planet and the Sun. Therefore, one needs to make a Galilean addition of velocities for the latter case that results in the swing-by maneuver. One can check that the elastic collision is satisfied, the energy gained by the spacecraft is equal to the energy lost by the planet, but this is so small that the planet's motion is not altered.[2] The reader can also look for other approaches to explain the slingshot effect, like derivations from special relativity,[4,7] classical Lagrangian[5] and work-impulse.[6] Our goal in this paper is to get some insights of how the gravity assist changes the course of a spacecraft away from the ecliptic plane like in Ulysses mission. Our motivation is that we can find no literature about 3D gravity assist at the introductory physics level.

Let us start with the description of our assumptions. We will use the word *planet* for the celestial body that alters the spacecraft's trajectory.

1. For simplicity, the Sun, the Earth and the planet are all in the same plane, the *ecliptic plane.* We also consider that the Sun's and the planet's equators lie in that plane.

2. The spacecraft motion can be divided in four parts, according to which celestial object contributes more to the gravitational force:

   (a) escaping the Earth,

   (b) moving by Sun interaction,

   (c) "swinging by" the planet and

(d) orbiting around the Sun or escaping the Solar System.

We concentrate our work on the last two parts, where the gravity assist effect is employed and causes the final trajectory.

3. After escaping the Earth, the spacecraft moves in a plane that is parallel to ecliptic.

4. The interaction between the planet and the spacecraft is considered as scattering phenomenon in the planet's frame, occurring in a very short time compared to the period of the planet's circular orbit. Therefore, the planet's motion is taken as a straight line during the slingshot effect.

II. HYPERBOLIC TRAJECTORY REVIEW

Our goal in this section is to determine the scattering angle of the spacecraft, β, due to the gravitational interaction with the planet, in the planet's frame. A hyperbolic trajectory is shown in Fig. 1. In polar coordinates, the equation for this kind of conic section is given by

$$r = \frac{a\left(\varepsilon^2 - 1\right)}{1 + \varepsilon \cos\theta}\,, \tag{1}$$

where $\varepsilon (> 1)$ is the eccentricity and a is the semi-major axis.

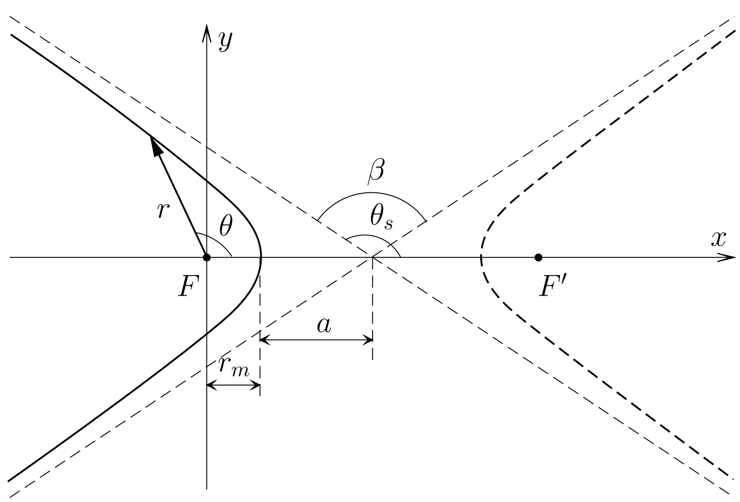

FIG. 1: Hyperbolic trajectory. F is the focal point where the force center is located.

As seen from this equation, a particular hyperbola is completely defined by two parameters: ε and a. In celestial mechanics, these parameters are usually written as functions of the first integrals of the motion: the energy, E, and the angular momentum, l, of a particle moving under the influence of a central inverse-square law force. In scattering problems, the impact parameter is used in place of the angular momentum.

We consider that the initial velocity u_i and the pericenter r_m are our known values, so that they will take the place of the above parameters, describing the spacecraft's hyperbolic orbit.

By solving for the eccentricity in Eq. (1) when $\theta = 0$ and $r = r_m$, we find

$$\varepsilon = 1 + \frac{r_m}{a}\,, \tag{2}$$

where the semi-major axis is obtained by the usual celestial-mechanics equation

$$a \equiv \frac{GMm}{2E} = \frac{GM}{u_i^2}\,, \tag{3}$$

since $E = \frac{1}{2}mu_i^2$, when the spacecraft is far away from central force source.

By taking $r \to \infty$ in the conic section equation (1) and solving for θ, we determine the angle between the asymptote and the x-axis

$$\cos\theta_s = -\frac{1}{\varepsilon}\,. \tag{4}$$

With the geometry shown in Fig. 1, the spacecraft's scattering angle is

$$\beta = 2\theta_s - \pi, \tag{5}$$

which has values $0 \leq \beta \leq \pi$ (attractive scattering).

III. THE SIMPLE 3D CASE FOR GRAVITY ASSIST

Let us first study the case when the spacecraft's-hyperbolic-orbit plane (SHOP) is perpendicular to the ecliptic plane (Fig. 2). Note that under assumption 4, SHOP is a fixed plane in the planet's frame, but it is a constant-velocity moving plane in the Sun's frame. For this particular case, the 3D vector analysis gets simplified and allows us to picture how the general case can be established. We also choose that the spacecraft flies over the planet's south pole. Our goal here is to determine the angle between the spacecraft's-final-orbit plane (SFOP) and the ecliptic plane, we will call it the *elevation angle*, γ.

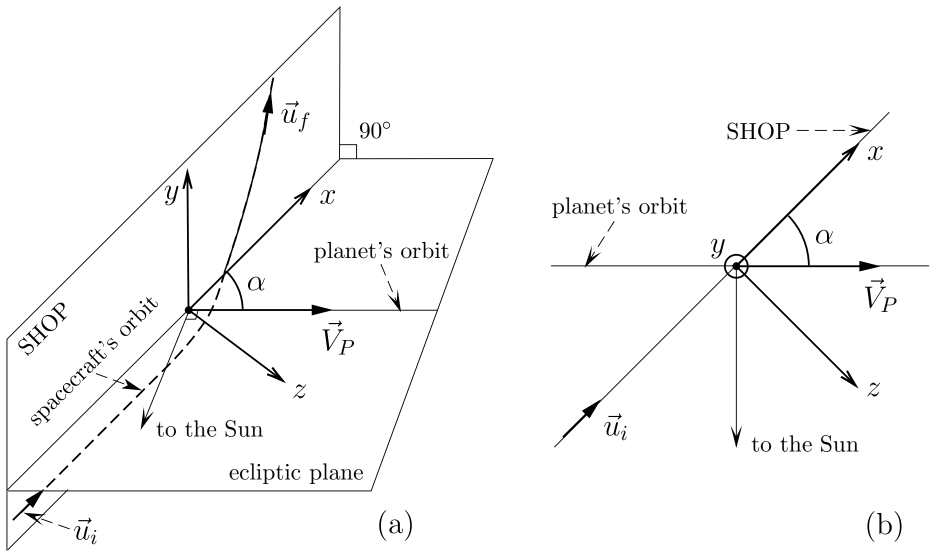

FIG. 2: The simple 3D case for gravity assist: (a) its visual representation in 2D and (b) the top view.

In a planet's frame as that shown in Fig. 2, the components of the planet's velocity relative to the Sun, $\vec{V}_P$, are

$$V_{P,x} = V_P \cos\alpha, \quad V_{P,y} = 0, \quad V_{P,z} = V_P \sin\alpha, \tag{6}$$

where α is the angle between the planet's velocity and the intersection of SHOP and the ecliptic plane. We consider $0 \le \alpha \le \pi$, meaning that the spacecraft is approaching a planet outside the Earth's orbit.

The initial and final velocities of the spacecraft in the Sun's frame, $\vec{v}_i$ and $\vec{v}_f$, are obtained by Galilean addition of velocities

$$\vec{v}_i = \vec{u}_i + \vec{V}_P, \quad \vec{v}_f = \vec{u}_f + \vec{V}_P, \tag{7}$$

where $\vec{V}_P$ obviously has the role of the velocity of the planet's frame with respect to the Sun's frame and $\vec{u}_i$ and $\vec{u}_f$ are the spacecraft's initial and final velocities in the planet's frame.

With the aid of Fig. 3, we find the magnitude v_i to be

$$v_i^2 = (u_i + V_{P,x})^2 + V_{P,y}^2 + V_{P,z}^2 = u_i^2 + V_P^2 + 2u_iV_P\cos\alpha. \tag{8}$$

Note that α is also the angle between $\vec{V}_P$ and $\vec{u}_i$.

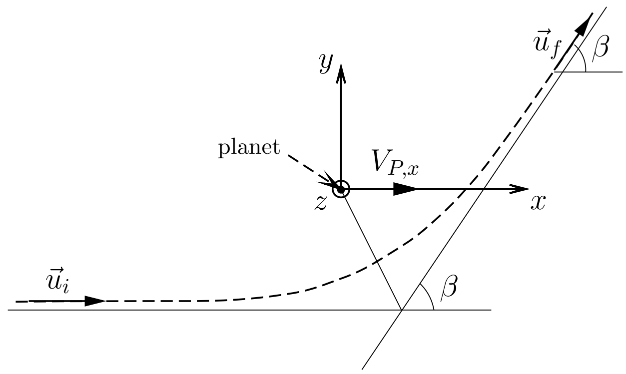

FIG. 3: The spacecraft's-hyperbolic-orbit plane (SHOP) and the spacecraft's motion as seen from the planet. The vector $\vec{V}_P$ lies in the zx-plane.

For the final velocity $\vec{v}_f$, we first compute the components of $\vec{u}_f$ in the planet's frame

$$u_{f,x} = u_f \cos\beta, \quad u_{f,y} = u_f \sin\beta. \tag{9}$$

Therefore, the magnitude v_f is given by

$$v_f^2 = (u_{f,x} + V_{P,x})^2 + (u_{f,y} + V_{P,y})^2 + V_{P,z}^2 = u_f^2 + V_P^2 + 2u_f V_P \cos\alpha\, \cos\beta. \tag{10}$$

Then, we subtract Eq. (8) from Eq. (10) and obtain

$$v_f^2 - v_i^2 = 2uV_P \cos\alpha\,(\cos\beta - 1), \tag{11}$$

since $u_i = u_f = u$ by assumption 4.

Eq. (11) allows us to distinguish three situations according to how initial and final velocities in the Sun's frame are related,

- $v_f > v_i$: The spacecraft increases its speed, if the planet scatters it ($\beta \neq 0$) and they encounter each other ($\frac{\pi}{2} < \alpha \leq \pi$),
- $v_f < v_i$: It slows down, if the planet scatters it ($\beta \neq 0$) and the spacecraft "tries" to catch the planet ($0 \leq \alpha < \frac{\pi}{2}$),
- $v_f = v_i$: There is no speed change, if there is no scattering ($\beta = 0$) or the spacecraft moves exactly perpendicular to the planet's motion ($\alpha = \frac{\pi}{2}$).

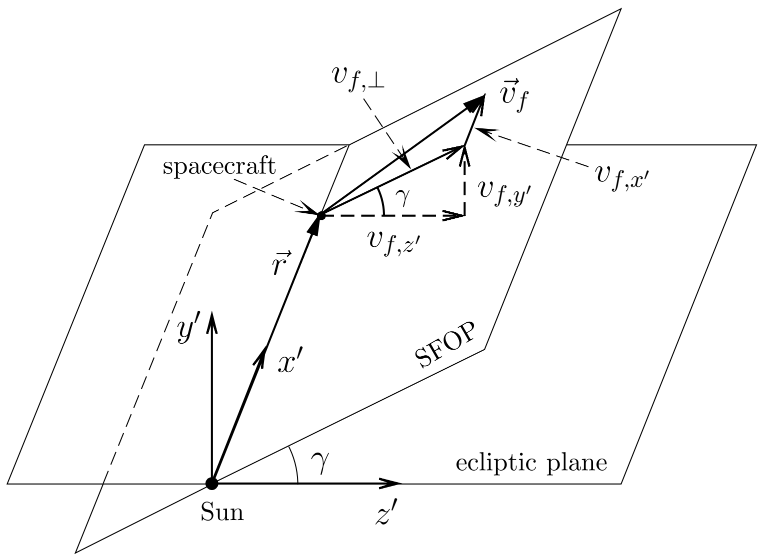

FIG. 4: The spacecraft's-final-orbit plane (SFOP) in the Sun's frame. $v_{f,\perp}$ is like the lever arm.

After the slingshot effect has occurred, we can determine the spacecraft's final orbit. According to assumption 2, the initial kinematic variables for the spacecraft orbiting around the Sun [part (d)] are those at the end of the slingshot [part (c)]. Thus, the initial velocity in part (d) is equal to $\vec{v}_f$ and the initial vector position $\vec{r}$ is taken to have a magnitude equal to the planet's distance from the Sun and lying in the ecliptic plane (Fig. 4). This approximation can be justified because the gravitational force due to the planet is much weaker than that due to the Sun (see a numerical calculation for Jupiter at the end of Sec. V).

From the angular momentum conservation and the properties of the cross product, the SFOP is completely defined by $\vec{v}_f$ and $\vec{r}$. Therefore, we conclude through Fig. 4 that

$$\tan\gamma = \frac{v_{f,y'}}{v_{f,z'}}\,, \tag{12}$$

where these velocity components are defined in the Sun's frame that has the x-axis aligned in the direction of the vector $\vec{r}$.

We recall Eq. (7) and use Fig. 5 to find

$$\tan\gamma = \frac{u_{f,y}}{V_P + u_{f,x}\cos\alpha} = \frac{\sin\beta}{(V_P/u) + \cos\alpha\,\cos\beta}\,. \tag{13}$$

Note that we have derived an expression that depends only on information before the slingshot.

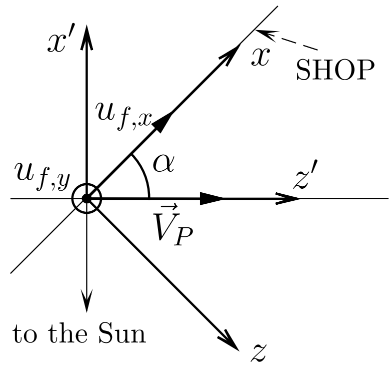

FIG. 5: Top view for the velocity components of the spacecraft in the Sun's frame.

IV. THE GENERAL 3D CASE FOR GRAVITY ASSIST

Let us now study the case when the angle between SHOP and the ecliptic plane is some value δ, not necessarily 90°. We repeat the same derivation procedure done in the above section, but including now the effect of the angle δ.

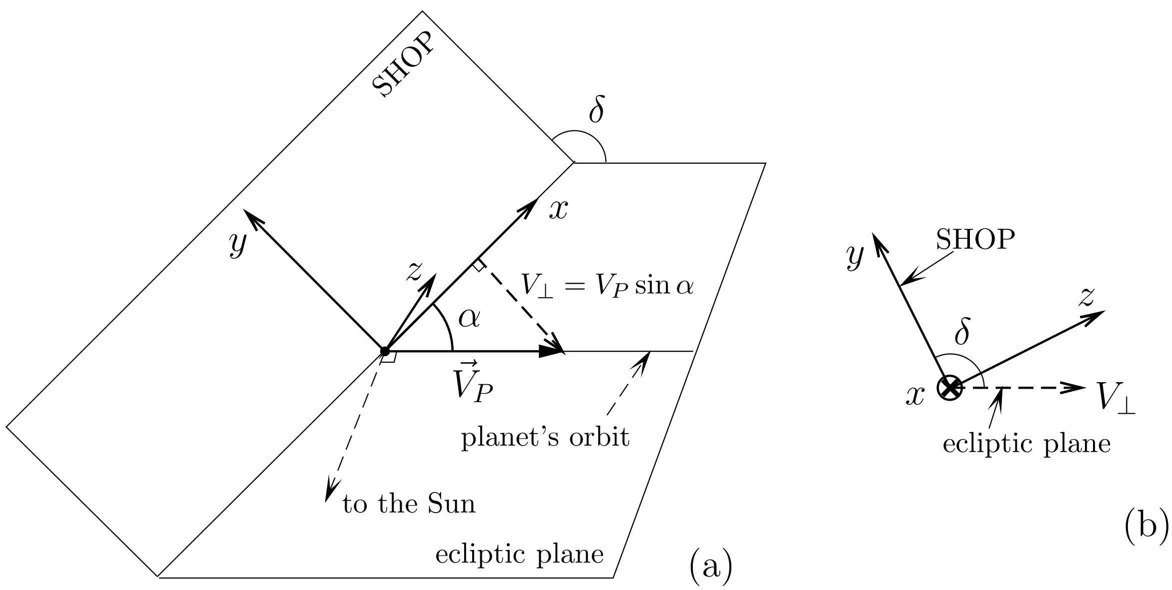

FIG. 6: The general slingshot case: (a) its visual representation in 2D and (b) the front view.

From Fig. 6, the components of $\vec{V}_P$ in the planet's frame are

$$V_{P,x} = V_P \cos\alpha, \quad V_{P,y} = V_P \sin\alpha \, \cos\delta, \quad V_{P,z} = V_P \sin\alpha \, \sin\delta, \tag{14}$$

where α is still the angle between the planet's velocity and the intersection of SHOP and the ecliptic plane.

Using Fig. 3 and Eq. (7), we find the magnitude v_i as we did before

$$v_i^2 = u_i^2 + V_P^2 + 2u_iV_P \cos\alpha. \tag{15}$$

Note that α still represents the angle between $\vec{V}_P$ and $\vec{u}_i$ and the velocity components of the spacecraft in the planet's frame, $\vec{u}_i$ and $\vec{u}_f$, do not change due to the SHOP orientation.

The magnitude v_f follows from Eqs. (9) and (14) in Eq. (7)

$$v_f^2 = u_f^2 + V_P^2 + 2u_fV_P(\cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta\,\cos\delta). \tag{16}$$

Then, we can subtract Eq. (15) from Eq. (16) and obtain

$$v_f^2 - v_i^2 = 2uV_P\left(\sin\alpha\,\sin\beta\,\cos\delta + \cos\alpha\,(\cos\beta - 1)\right), \tag{17}$$

since $u_i = u_f = u$. With this result, the reader can study the ranges of α, β and δ that make the spacecraft increase or decrease its speed or even those that lead to no speed change. She can also verify the usual 2D case ($\delta = 0, \pi$).[1,2,3]

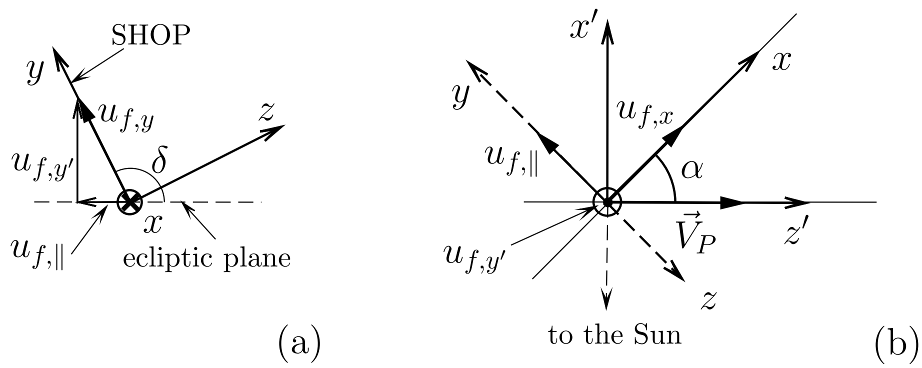

FIG. 7: The velocity components in the Sun's frame: (a) front view and (b) top view.

From the Fig. 7 and Eq. (12), the elevation angle is given by

$$\begin{aligned}\tan\gamma = \frac{v_{f,y'}}{v_{f,z'}} &= \frac{u_{f,y}\sin\delta}{V_P + u_{f,x}\cos\alpha - u_{f,y}\cos\delta\,\cos(\pi/2+\alpha)} \\ &= \frac{\sin\beta\,\sin\delta}{(V_P/u) + \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta\,\cos\delta}\,.\end{aligned} \tag{18}$$

V. NUMERICAL CALCULATIONS

Let us compute the elevation angle using Ulysses spacecraft data. From any college physics textbook,[9] we have the following Jupiter information:

$$M_J = 1.90 \times 10^{27}\ \text{kg}, \quad a_J = 7.78 \times 10^{11}\ \text{m}, \quad R_J = 6.99 \times 10^{7}\ \text{m}, \quad \tau_J = 3.74 \times 10^{8}\ \text{s},$$

where they are its mass, its distance from the Sun, its radius and its period, respectively.

From Ulysses web page,[8] we have the spacecraft information:

$$m = 366.7\ \text{kg}, \quad u = u_i = 13.896\ \text{km/s}, \quad v_i = 16.184\ \text{km/s}, \quad r_m = 6.3\, R_J.$$

Therefore, the equations of Sec. II result in ($G = 6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2$)

$$a = 6.56 \times 10^{8}\ \text{m}, \quad \varepsilon = 1.67, \quad \theta_s = 127^\circ, \quad \beta = 74^\circ.$$

Assuming uniform circular motion, we estimate Jupiter's speed, V_P, and use Eq. (15) to get α,

$$V_P = \left(\frac{2\pi}{\tau_J}\right) a_J = 13.1\ \text{km/s}, \quad \alpha = 106^\circ.$$

We let the angle δ be a free parameter. Table I shows the final speed and the elevation angle computed from Eqs. (16) and (18) with the above data. The usual condition for elliptic orbit $E < 0$ can be rewritten as $v < \sqrt{2GM_S/r} = 18.5$ km/s, with $r = a_j$ as it was considered at the end of Sec. III for deriving γ ($M_S = 1.99 \times 10^{30}$ kg).

From Table I, we conclude that the minimum elevation angle that would keep the spacecraft orbiting around the Sun is around 48°, meaning that the spacecraft would fly over a Jupiter's pole. Lower values of γ would imply that the spacecraft will escape the Solar System. Notice that δ represents the maximum latitude (both north and south) that SHOP crosses in the planet, under the assumption 1 (for $\delta > 90^\circ$, the related parallel is $180^\circ - \delta$). By symmetry, one could infer the results for $180^\circ < \delta < 360^\circ$. When the spacecraft encounters the planet and reaches the latitude 20.3° S as maximum (no further south), its final orbit around the Sun will be exactly perpendicular to the ecliptic plane. This situation repeats when it "tries" to catch the planet and reaches the latitude 20.3° N as maximum (no further north). The maximum speed given by gravity assist is obtained when the spacecraft "chases" the planet on the ecliptic plane and the minimum when they encounter each other on the ecliptic plane.

δ (deg)	v_f (km/s)	γ (deg)
0	26.0	0.0
15	25.7	8.0
30	25.1	16.1
45	24.0	24.1
60	22.5	32.1
90	18.4	48.0
120	13.0	64.1
146.9	7.4	80.0
150	6.8	82.1
159.7	4.6	90.0
165	3.5	95.9
170	2.4	104.5
175	1.4	122.7
180	0.8	180.0

TABLE I: The final speed and the elevation angle as functions of δ.

Let us check the agreement between our results and the actual Ulysses's orbit data. The Ulysses's elevation angle is around 80°, then from Table I, $v = 7.4$ km/s, which implies that its semi-major axis is about 3.10 AU (we have used Eq. (3) with $E/m = v^2/2 - GM_S/a_J$). This conclusion is quite close with the actual Ulysses's semi-major axis 3.37 AU (computed by using the first aphelion 5.40 AU and the first perihelion 1.34 AU, obtained from the web page).[8] This comparison shows that our 3D slingshot formulation is numerically acceptable in spite of our assumptions and approximations. Moreover, it provides physical insight into how the gravity assist is used to deviate the spacecraft not only on the ecliptic plane but also away from it, according to the navigator's desire.

Finally, let us numerically justify our approximation in Sec. III for Jupiter case. The Sun's gravitational force around Jupiter is stronger than that of Jupiter when objects are located at distances of $r \gtrsim \sqrt{M_J/M_S}\, a_J = 0.03\, a_J = 334\, R_J$, so that our claim in Sec. III is fairly good.

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[8] Ulysses web pages: `<http://ulysses.jpl.nasa.gov/>`, `<http://ulysses-ops.jpl.esa.int/>` The initial speeds u_i and v_i are chosen from the ESA-JPL file: `weekbody.txt`, when Ulysses had its minimal speed in Jupiter's frame (date: December 16, 1991), as the frontier separating parts (b) and (c) of assumption 2. Strictly, m is the total mass at launch.

[9] See, for example, R. A. Serway and J. W. Jewett, *Physics for scientists and engineers*, 7th ed. (Thomson-Brooks/Cole, Belmont, CA, 2008).